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The Prime Spectrum of a Leavitt Path Algebra

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The Prime Spectrum of a Leavitt Path Algebra

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by

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ABSTRACT

The Prime Spectrum of a Leavitt Path Algebra

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Naomi Lee Burkhart

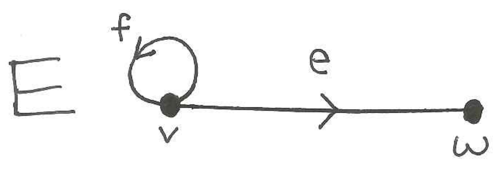
Given an arbitrary directed graph E and field K, we study the partially ordered set $Spec(L_K(E))$ of prime ideals of the corresponding Leavitt path algebra $L_K(E)$, partially ordered by set inclusion. We work towards classifying which posets appear in this way, for general graphs E, row-finite graphs E, and finite graphs E. By considering special sets of vertices (namely, the maximal tails) we are able to determine the structure of the prime ideals, and prove that every countable poset that has the DCC and locally has the ACC is both of the form $Spec(L_K(E))$ and $Spec_\gamma(L_K(E))$ for some countable, row-finite graph E and for an arbitrary field K, where $Spec_\gamma(L_K(E))$ is the collection of prime, graded ideals of $L_K(E)$. If we only look at the finite graphs E, by doing explicit constructions we find that the posets that arise for $Spec_\gamma(L_K(E))$ are precisely the finite posets, the posets that arise for $Spec(L_K(E))$ are those finite posets, with sets of infinite cardinality $max\{\aleph_0, |K|\}$ inserted at arbitrary locations throughout the poset, and that the posets that arise for $Spec(L_K(E))\setminus Spec_\gamma(L_K(E))$ look like the finite posets, except that each point is replaced with infinitely many non-comparable ones.

1. Preliminaries

A Leavitt path algebra is a type of algebra which is built from a directed graph. Although the notion of constructing an algebraic structure from a geometric one is not new, as of recently there has been a peak in interest in this particular construction. In the last several decades discoveries regarding Leavitt path algebras have been made which have sparked interest in the subject, including deep connections between them and various other topics, such as non-stable K-theory and C^*-algebras. For instance: from a directed graph one is able to construct a C^*-algebra – called a graph C^*-algebra – and there are numerous parallels between these and the Leavitt path algebra constructed from the same graph. Results on one side have forecasted what type of corresponding result might occur on the other side, and many of these predictions have been accurate. There have even been some instances where the development of a result for Leavitt path algebras has provided the means to get the parallel C^* result.

Additionally, many of the standard examples of rings show up as Leavitt path algebras, including $K[x, x^{-1}]$, $M_n(K)$, and $M_{\mathbb{N}}(K)$ for an arbitrary field K. So also do the Leavitt algebras, which are algebras invented by Leavitt to help provide examples of finitely generated free modules whose ranks exhibit all possible behaviors that one could imagine.

We begin by introducing some definitions and basic theory. A *directed graph* is a tuple $E = (E^0, E^1, r, s)$ where E^0, E^1 are disjoint sets and $r, s : E^1 \to E^0$ are maps. We refer to the elements of E^0 as *vertices*, and the elements of E^1 as *edges*. Given an edge e, we call $s(e)$ the source of e and $r(e)$ the range of e. For example, we can construct a directed graph E with vertices $E^0 = \{v, w\}$, and edges $E^1 = \{e, f\}$ such that $s(e) = s(f) = r(f) = v$, $r(e) = w$, which might be depicted as follows:



The above graph is known as the *Toeplitz graph.*

For the purposes of this paper, only graphs which are directed are of interest, so we will drop "directed" and simply use "graph" to indicate a directed graph.

A *path* in a graph E is a sequence $\rho = e_1 \cdots e_n$ such that $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$ for all i. We say ρ starts at $s(\rho) := s(e_1)$ and ends at $r(\rho) := r(e_n)$. In the case that $n = 0$, ρ is just a single vertex $\rho = v$ and we set $s(\rho) = v = r(\rho)$. For vertices $v, w \in E^0$ we write $v \geq w$ in case there exists a path in E from v to w.

For E as in the previous image, we have, for instance, the paths v, e, $fffe$, and $v \geq w$ but $w \not\geq v$.

For the remainder of the paper we fix E to be a graph and K to be a field.

For a path $\rho = e_1 \cdots e_n$, we define $\rho^0 = \{s(e_1), s(e_2), ..., s(e_n), r(e_n)\}$ and $\rho^1 = \{e_1, ..., e_n\}$. The *length* of ρ is n. If the length of ρ at least 1 and $s(\rho) = v = r(\rho)$, we call ρ a *closed path based at* v. If additionally $s(e_i) \neq v$ for all $i \neq 1$, we call ρ *simple.*

A closed path $\rho = e_1 \cdots e_n$ based at v is called a *cycle based at* v if $s(e_i) \neq s(e_j)$ for all $i \neq j$. Let ρ be a cycle based at v, and for each $w \in \rho^0$, let ρ_w be the cycle based at w with $\rho^1 = \rho^1_w$. Then $c = \{\rho_w : w \in \rho^0\}$ is a *cycle* and we define $c^0 = \rho^0$, $c^1 = \rho^1$. Given such a cycle c and a vertex $w \in c^0$, we denote by c_w the element of c which is based at w. Notice $c = \{c_w : w \in c^0\}$. Finally, a *loop* is a closed path of length 1.

Given a graph E and a path ρ in E, an *exit* for ρ is an edge e such that $s(e) = s(f)$ for some $f \in \rho^1$, and $e \neq f$. We will see shortly that the cycles of a graph and their exits play a fundamental structural role in the corresponding Leavitt path algebra.

Given a graph E, the *dual graph* of E is the graph $E^* = (E^0, (E^1)^*, r^*, s^*)$ where $(E^1)^* = \{e^* : e \in E^1\}$ with $e^* \neq f^*$ when $e \neq f$, and for each $e \in E^1$, $r^*(e^*) = s(e)$, $s^*(e^*) = r(e)$. The edges $e \in E^1$ are called *real edges* and edges $e^* \in (E^1)^*$ are called *ghost edges.*

We now define the object at the center of our focus:

Let E be a graph and K a field. The *Leavitt path algebra* of E over K, written $L_K(E)$, is the K-algebra presented by the set $E^0 \sqcup E^1 \sqcup (E^1)^*$ with the following relations:

For all $v, v' \in E^0$ and $e, e' \in E^1$:

(i) $vv' = \delta_{v,v'}v$

(ii) $e = s(e)e = er(e)$

(iii) $e^* = r(e)e^* = e^*s(e)$

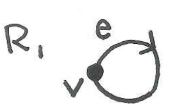
(iv) $e^*e' = \delta_{e,e'}r(e)$

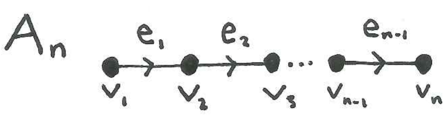
and for all $w \in E^0$ with $0 < |s^{-1}(w)| < \infty$,

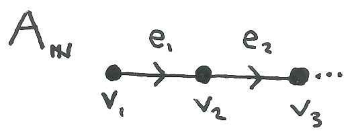
(v) $w = \sum_{e \in E^1, s(e)=w} ee^*$

In general one uses the same symbols for elements of E and corresponding elements of $L_K(E)$. Note that there is no ambiguity , since the map $E^0 \sqcup E^1 \sqcup (E^1)^* \to L_K(E)$ is injective, by [6, Lemma 1.6].

For







we have $L_K(R_1) \cong K[x, x^{-1}]$ by the isomorphism sending v to 1, e to x, and e^* to x^{-1}. We have $L_K(A_n) \cong M_n(K)$ via $f : L_K(A_n) \to M_n(K)$ with $f(v_i) = E_{i,i}$, $f(e_i) = E_{i,i+1}$, and $f(e_i^*) = E_{i+1,i}$, where $E_{i,j}$ is the matrix with a 1 in its (i,j)th position and 0's elsewhere. Lastly $L_K(A_{\mathbb{N}}) \cong M_{\mathbb{N}}(K)$ via $f' : L_K(A_{\mathbb{N}}) \to M_{\mathbb{N}}(K)$ with

$f'(v_i) = E_{i,i}$, $f'(e_i) = E_{i,i+1}$, and $f'(e_i^*) = E_{i+1,i}$. (Here $M_{\mathbb{N}}(K)$ is the collection of $\mathbb{N} \times \mathbb{N}$ matrices over K with at most finitely many nonzero elements.)

Note that in general $L_K(E)$ is not unital, and that it is unital precisely when E^0 is finite (in which case the identity is the sum over the elements of E^0).

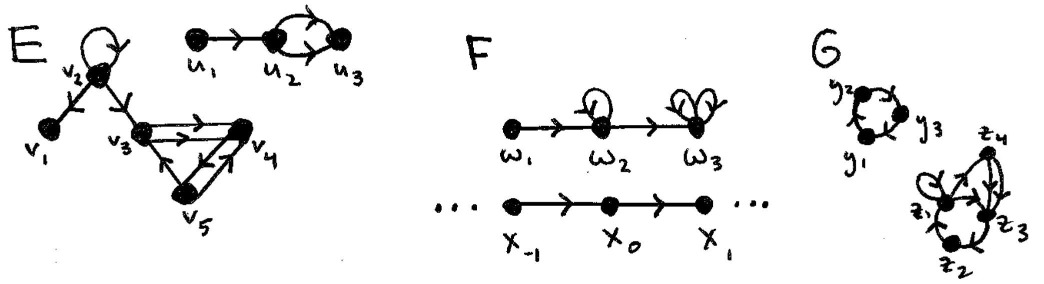
A vertex $v \in E^0$ is a *sink* in case $s^{-1}(v)$ is empty and is an *infinite emitter* in case $s^{-1}(v)$ is infinite. A graph E is called *row-finite* in case it has no infinite emitters. Due to relation (v) in the above definition, we see how it is plausible that the requirement for a graph E to be row-finite may greatly simplify the theory of Leavitt path algebras. This is, indeed, the case, so many authors choose to restrict their attention to only those graphs which are row-finite. We will also do this here, for the majority of the results.

We now turn our discussion to certain subsets of E^0 which, as we will see shortly, play a key role in determining the structure of $L_K(E)$.

A subset $H \subseteq E^0$ is *saturated* if whenever $\emptyset \neq r(s^{-1}(v)) \subseteq H$, for $v \in E^0$, we have $v \in H$, and is *hereditary* if whenever $v \in H$ and $v \geq w$, then also $w \in H$. Set

$$\mathcal{H}_E := \{H \subseteq E^0 : H \text{ is saturated and hereditary}\}.$$

The graphs E, F, G below have the following saturated, hereditary sets of vertices:



$$\mathcal{H}_E = \{\emptyset, \{v_1\}, \{v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4, v_5\}, \{u_1, u_2, u_3\}, \{u_1, u_2, u_3, v_1\}, \{u_1, u_2, u_3, v_3, v_4, v_5\}, \{u_1, u_2, u_3, v_1, v_3, v_4, v_5\}, E^0\}$$

$$\mathcal{H}_F = \{\emptyset, \{w_3\}, \{w_1, w_2, w_3\}, \{x_i : i \in \mathbb{Z}\}, \{w_3\} \cup \{x_i : i \in \mathbb{Z}\}, F^0\}$$

$$\mathcal{H}_G = \{\emptyset, \{y_1, y_2, y_3\}, \{z_1, z_2, z_3, z_4\}, G^0\}$$

Note that $L_K(E)$ has a $\mathbb{Z}$-grading induced by setting $deg(v) = 0$, $deg(e) = 1$, and $deg(e^*) = -1$ for each $v \in E^0$, $e \in E^1$. The degree of a real path ρ is then $length(\rho)$, and the degree of a ghost path ρ is $-length(\rho)$.

In any $\mathbb{Z}$-graded ring, the ideals which are generated by homogeneous elements are graded. In particular, for all $X \subseteq E^0$, we know $\langle X \rangle$ is graded. In the case of Leavitt path algebras, however, all graded ideals arise in this way.

Lemma 1.1. ([2, Lemma 2.4.3]) *Let I be an ideal of $L_K(E)$. Then $I \cap E^0 \in \mathcal{H}_E$.*

Proposition 1.2. ([2, Theorem 2.5.9]) *Let E be row-finite and let G denote the lattice of graded ideals of $L_K(E)$. The following map provides a lattice isomorphism*

$$\varphi : G \to \mathcal{H}_E \qquad \varphi(I) = I \cap E^0$$

with inverse given by

$$\varphi' : \mathcal{H}_E \to G \qquad \varphi'(H) = \langle H \rangle.$$

The previous result appears originally as [3, Theorem 5.3].

We also investigate the prime ideals of $L_K(E)$, which are deeply connected to another type of special set of vertices: the maximal tails. A nonempty subset $M \subseteq E^0$ is called a *maximal tail* if it satisfies the following three conditions:

(MT1) $w \in M$, $v \geq w \Rightarrow v \in M$

(MT2) $w \in M$, $s^{-1}(w) \neq \emptyset \Rightarrow r(s^{-1}(w)) \cap M \neq \emptyset$

(MT3) $x, y \in M \Rightarrow \exists w \in M$ such that $x \geq w$, $y \geq w$.

Notice that M satisfies (MT1) if and only if $E^0\backslash M$ is hereditary and M satisfies (MT2) if and only if $E^0\backslash M$ is saturated. We record this in the folowing lemma, which we will use regularly and often without reference:

Lemma 1.3. ([5, Lemma 2.1]) *A subset $M \subseteq E^0$ satisfies Conditions (MT1) and (MT2) if and only if $E^0\backslash M \in \mathcal{H}_E$.*

We will also need the following, which is originally stated as [4, Proposition 5.6].

Proposition 1.4. ([2, Proposition 4.1.4]) *Let E be row-finite and let $H \in \mathcal{H}_E$. The following are equivalent*

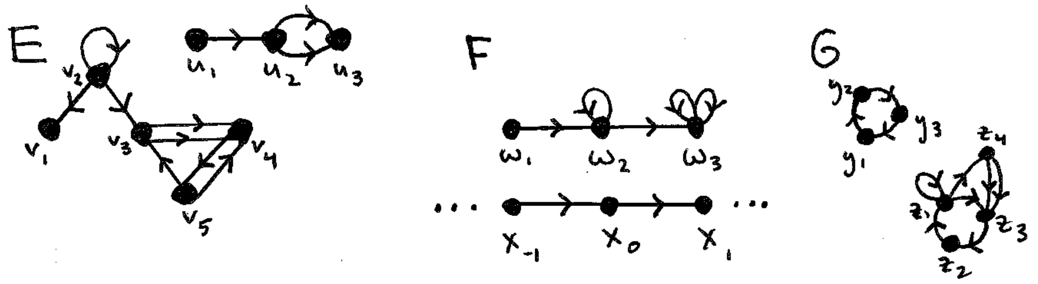
(a) *$\langle H \rangle$ is prime*
(b) *$M = E^0\backslash H$ is a maximal tail*
(c) *M is downward directed*

Here by *downward directed* we mean M satisfies (MT3).

Given a maximal tail M and a path ρ, we say ρ is a *path in M* if $\rho^0 \subseteq M$. We say a cycle c has an *exit in M* if it has an exit e such that $r(e) \in M$. We set

$M(E) :=$ {maximal tails in E}
$M_\gamma(E) := \{M \in M(E) :$ every simple closed path in M has an exit in $M\}$
$M_\tau(E) := M(E)\backslash M_\gamma(E)$

For the graphs E, F, G below, we have the following maximal tails:



$$M(E) = \{\{v_1, v_2\}, \{v_2, v_3, v_4, v_5\}, \{u_1, u_2, u_3\}, \{v_2\}\}$$
$$M_\gamma(E) = \{\{v_1, v_2\}, \{v_2, v_3, v_4, v_5\}, \{u_1, u_2, u_3\}\}, \quad M_\tau(E) = \{\{v_2\}\}$$
$$M(F) = \{\{w_1, w_2, w_3\}, \{x_i : i \in \mathbb{Z}\}, \{w_1, w_2\}\}$$
$$M_\gamma(F) = \{\{w_1, w_2, w_3\}, \{x_i : i \in \mathbb{Z}\}, \}, \quad M_\tau(F) = \{\{w_1, w_2\}\}$$
$$M(G) = \{\{z_1, z_2, z_3, z_4\}, \{y_1, y_2, y_3\}\}$$
$$M_\gamma(G) = \{\{z_1, z_2, z_3, z_4\}\}, \quad M_\tau(G) = \{\{y_1, y_2, y_3\}\}$$

Lastly, we set

$$\begin{aligned}
&Spec(L_K(E)) := \{\text{prime ideals of } L_K(E)\} \\
&Spec_\gamma(L_K(E)) := \{\text{prime, graded ideals of } L_K(E)\} \\
&Spec_\tau(L_K(E)) := Spec(L_K(E)) \backslash Spec_\gamma(L_K(E)).
\end{aligned}$$

Proposition 1.5. ([5, Lemma 2.7]) *Let $I \in Spec_\tau(L_K(E))$, and let $H = I \cap E^0$. Then $\langle H \rangle$ is prime, and $E^0 \backslash H \in M_\tau(E)$.*

Corollary 1.6. *Let E be row-finite. For each prime ideal I of $L_K(E)$, there is a unique largest prime, graded ideal contained in I.*

Proof. Pick $I \in Spec(L_K(E))$ which is not graded. Put $H = I \cap E^0$. By Proposition 1.5 we have $\langle H \rangle \in Spec_\gamma(L_K(E))$. Then $\langle H \rangle$ is the largest graded ideal contained in I, for if it is not then there is a graded ideal $I' \subseteq I$ containing some $v \in E^0 \backslash H$. But no such v lies in I so this is impossible. □

Here we record some miscellaneous results which we will need later.

Lemma 1.7. ([2, Lemma 2.2.7]) *Let E be an arbitrary graph. If c is a cycle without exits based at a vertex v, then*

$$vL_K(E)v = \left\{ \sum_{i=m}^{n} k_i c^i : k_i \in K,\ m \leq n,\ m, n \in \mathbb{Z} \right\} \cong K[x, x^{-1}],$$

via an isomorphism that sends v to 1, c to x, and c^* to x^{-1}.

If $H \subseteq E^0$ is hereditary, we define the *quotient graph*

$$E/H := (E^0 \backslash H,\ \{e \in E^1 : r(e) \notin H\},\ r|_{(E/H)^1},\ s|_{(E/H)^1}).$$

Lemma 1.8. ([2, Corollary 2.4.13]) *Let E be a row-finite graph and let K be any field. If $H \in \mathcal{H}_E$, then $L_K(E)/\langle H\rangle \cong L_K(E/H)$ as $\mathbb{Z}$-graded K-algebras.*

Lemma 1.9. ([2, Corollary 2.8.17]) *Let E be an arbitrary graph and K any field. If I and J are arbitrary ideals of $L_K(E)$ then $IJ = JI$.*

2. The Stucture of the Prime Ideals

Before attempting to answer the primary question of the thesis, we quote some results regarding $Spec(L_K(E))$, $Spec_\gamma(L_K(E))$, and $Spec_\tau(L_K(E))$.

From [2], page 109 we have the following key lemma, which we will use frequently:

Lemma 2.1. *For any row-finite graph E and field K the map*

$$\Psi : Spec_\gamma(L_K(E)) \to M(E) \qquad \Psi(I) = E^0 \backslash I$$

is a bijection, with inverse

$$\Psi' : M(E) \to Spec_\gamma(L_K(E)) \qquad \Psi'(M) = \langle E^0 \backslash M\rangle.$$

Corollary 2.2. *For any row-finite graph E and field K*

$$(Spec_\gamma(L_K(E)), \subseteq) \cong (M(E), \supseteq).$$

Corollary 2.3. *For any row-finite graph E and field K*

$$Spec_\gamma(L_K(E)) = \{\langle E^0 \backslash M\rangle : M \in M(E)\}.$$

Lemma 2.4. *Let E be row-finite. If $I \in Spec(L_K(E))$, then $E^0 \backslash I$ is a maximal tail.*

Proof. If I is graded the result follows from Lemma 2.1, and if I is not graded it follows from Proposition 1.5. $\square$

If $H \in \mathcal{H}_E$, we define $C_H := \{c : c$ is a cycle in E with $c^0 \cap H = \emptyset$, and so that for each exit e of c, $r(e) \in H\}$. For a polynomial $p(x) \in K[x]$ and a cycle c, we define $p(c) = \{p(c_v) : v \in c^0\}$, where evaluating x^0 at c_v gives v, and evaluating x^{-n}, $n > 0$, at c_v gives $(c_v^*)^n$. Then given $H \in \mathcal{H}_E$, $C \subseteq C_H$, and a set of polynomials $P = \{p_c(x) : c \in C\}$, we define $P_C := \cup_{c \in C} p_c(c)$.

Set

$$\mathcal{P}(K) := \{p(x) = 1 + k_1 x + \cdots + k_n x^n : p(x) \in K[x] \backslash K\}$$

and

$$\dot{\mathcal{P}}(K) := \{p(x) \in \mathcal{P}(K) : p(x) \text{ is irreducible}\}.$$

If the field K is clear, we will denote $\mathcal{P}(K), \dot{\mathcal{P}}(K)$ by simply $\mathcal{P}, \dot{\mathcal{P}}$, respectively.

Proposition 2.5. ([2, Proposition 2.8.5]) *Let I be an ideal of $L_K(E)$ and put $H = I \cap E^0$. Let J denote $I/\langle H \rangle$, which we may view as an ideal of the Leavitt path algebra of the quotient graph $L_K(E/H)$ by Lemma 1.8. Then there exists a set $C \subseteq C_H$ and a set $P = \{p_c(x) : c \in C\} \subseteq \mathcal{P}(K)$ such that $J = \oplus_{c \in C} \langle p_c(c) \rangle$. Further, the sets C and P are uniquely determined by I.*

A combination of [2], Theorems 2.8.10 and 2.8.11 yields the following result, which plays a central role in the upcoming chapters.

Theorem 2.6. *Let E be a row-finite graph and K any field. Set $\mathcal{D} = \{(H, C, P) : H \in \mathcal{H}_E,\ C \subseteq C_H,\ P \subseteq \mathcal{P}(K),\ |C| = |P|\}$ and $\mathcal{I} = \{$ideals of $L_K(E)\}$. Then the following map is a bijection:*

$$\varphi : \mathcal{D} \to \mathcal{I} \qquad \varphi(H, C, P) = \langle H \cup P_C \rangle$$

with inverse given by

$$\varphi' : \mathcal{I} \to \mathcal{D} \qquad \varphi'(I) = (H, C, P)$$

where $H = I \cap E^0$ and C and P are as described in Proposition 2.5.

Corollary 2.7. *Let E be row-finite. For each $M \in M_\gamma(E)$ there is a unique ideal I of $L_K(E)$ for which $I \cap E^0 = E^0 \backslash M$. Namely, $I = \langle E^0 \backslash M \rangle \in Spec_\gamma(L_K(E))$.*

Proof. For $H = E^0 \backslash M$, notice that $H \in \mathcal{H}_E$ by Lemma 1.3 and $C_H = \emptyset$. Hence, by Theorem 2.6, $I = \langle H \rangle$ is the unique ideal which satisfies $I \cap E^0 = H$. Moreover $I \in Spec_\gamma(L_K(E))$ by Corollary 2.3. $\square$

For $M \in M_\tau(E)$, there are infinitely many ideals I of $L_K(E)$ for which $I \cap E^0 = E^0 \backslash M$, by the following key result.

Theorem 2.8. ([2, Theorem 4.1.8]) *Let E be a row-finite graph and K any field. Then there is a bijection*

$$Spec_\tau(L_K(E)) \to M_\tau(E) \times MaxSpec(K[x, x^{-1}])$$

given by

$$I \mapsto (E^0 \backslash I, \langle P \rangle)$$

where P is as in Proposition 2.5.

The inverse of this bijection is the map

$$M_\tau(E) \times MaxSpec(K[x, x^{-1}]) \to Spec_\tau(L_K(E))$$

given by

$$(M, \mathfrak{m}) \mapsto \langle (E^0 \backslash M) \cup p(c) \rangle,$$

where p is the unique polynomial in $\dot{\mathcal{P}}(K)$ that generates $\mathfrak{m}$ and c is the only cycle in M which has no exit in M.

This result is originally found in [5], although it appears there without an explicit description of the bijection. After the research efforts for this paper had begun the result reappeared in a more recent release of [2], this time with the maps written out (as in the version quoted above). Nevertheless, I have recorded my own version of the result here, in addition to its proof, which will come later in Chapter 5 after the development of some theory.

Theorem 2.9. *Let E be a row-finite graph and let K be a field. The map*

$$\Lambda : M_\tau(E) \times \dot{\mathcal{P}}(K) \to Spec_\tau(L_K(E))$$

$$(M, p) \mapsto \langle (E^0 \backslash M) \cup p(c_{E^0\backslash M}) \rangle$$

is a bijection, where $c_{E^0\backslash M}$ *is the unique cycle in* $C_{E^0\backslash M}$.

The previous two theorems are related by the following standard result.

Proposition 2.10. *The map*

$$\tau : \dot{\mathcal{P}}(K) \to MaxSpec(K[x, x^{-1}]) \qquad \tau(p) = \langle p \rangle$$

is a bijection.

Notice that normalizing the elements of $\dot{\mathcal{P}}(K)$ to all have constant term one gives us injectivity of τ.

3. The Target

When considering the Leavitt path algebra of a graph E over a field K one might ask what posets appear for $Spec(L_K(E))$, $Spec_\gamma(L_K(E))$, or even $Spec_\tau(L_K(E))$, each partially ordered by set inclusion. The main goal of this thesis is to answer this question. Although we do not succeed completely, we are able to give a partial answer, and even give a complete answer if we restrict our attention to those graphs which are finite.

In the next chapter we develop some theory regarding the sets of vertices of the form $I \cap E^0$ for an ideal I. As we have already seen (Lemma 2.1), in the case of $Spec_\gamma(L_K(E))$ it suffices to understand these sets – or rather, their complements (the maximal tails). As we will see later, in the case of $Spec(L_K(E))$ and $Spec_\tau(L_K(E))$ it will suffice to understand the possible structures of both $M(E)$ and $M_\tau(E)$.

In Chapter 5 we direct our attention towards the non-graded, prime ideals. After giving some results in regards to their structure, we are able to prove Theorem 5.4 which allows us to write out $Spec_\tau(L_K(E))$ explicitly. We then prove some lemmas concerning the inclusion relations between the non-graded, prime ideals and the other prime ideals.

In Chapter 6 we show that any countable poset that has the DCC and locally has the ACC shows up as both $Spec(L_K(E))$ and $Spec_\gamma(L_K(E))$ for a countable, row-finite graph E and for an arbitrary field K. In particular, this shows that Leavitt path algebras provide examples of rings whose set of (graded) prime ideals are of arbitrary finite size, and with arbitrary inclusion relations.

In the final chapter we answer the version of our question where "graph" is replaced with "finite graph". We show that the posets of the form $Spec_\gamma(L_K(E))$ for some finite graph E are precisely the finite posets, that the posets of the form $Spec(L_K(E))$ are of a special type, and are obtained from $Spec_\gamma(L_K(E))$ by inserting identical pieces at arbitrary places throughout the poset, and that the posets of the form $Spec_\tau(L_K(E))$ are made of just those identical pieces.

Note that from ring theory, we have the following well known constraints:

Lemma 3.1. *Every nonempty chain in $Spec(L_K(E))$ has a greatest lower bound in $Spec(L_K(E))$, namely the intersection of the chain.*

Lemma 3.2. *Let E be row-finite. Every nonempty chain in $Spec_\gamma(L_K(E))$ has a greatest lower bound in $Spec_\gamma(L_K(E))$.*

Also note that whichever posets arise for $Spec_\gamma(L_K(E))$ for a row-finite graph E also arise for $Spec(L_K(E))$:

Proposition 3.3. *Let E be row-finite. There exists a row-finite graph F for which $(Spec_\gamma(L_K(E)), \subseteq) \cong (Spec(L_K(F)), \subseteq)$.*

Proof. By Corollary 2.3, $Spec_\gamma(L_K(E)) = \{\langle E^0 \backslash M\rangle : M \in M(E)\}$. Construct F from E by adding a loop to each vertex $v \in E^0$ which lies along a cycle. Then notice $M(E) = M(F)$ and $M(F) = M_\gamma(F)$. By Corollary 2.2, the first equality gives us $Spec_\gamma(L_K(E)) \cong Spec_\gamma(L_K(F))$, and the second gives us $M_\tau(F) = \emptyset$. Thus, by Theorem 2.9, $Spec_\tau(L_K(F)) = \emptyset$ and so

$$Spec(L_K(F)) = Spec_\gamma(L_K(F)) \cong Spec_\gamma(L_K(E)). \qquad \square$$

Additionally, when determining which posets appear for $Spec_\gamma(L_K(E))$ it suffices to consider graphs whose cycles are of length 1:

Lemma 3.4. *Let $E = (E^0, E^1, r, s)$ be a row-finite graph and let $F = (F^0, F^1, r_F, s_F)$ be the graph obtained from E by contracting every cycle down to a cycle of length 1. Formally: Let $\{c_i : i \in I\}$ be the set of distinct cycles in E. We form an equivalence relation on this set by taking the transitive closure of the relation $\sim$, where $c_i \sim c_j$ if and only if $c_i^0 \cap c_j^0 \neq \emptyset$. Let $\{[c_i] : i \in J\}$ be the collection of distinct equivalence classes, and for each $i \in J$ let α_i be the number of cycles c_j for which $[c_j] = [c_i]$. Then*

$$F^0 = \{v_j : j \in J\} \cup \left(E^0 \setminus \cup_{i \in I} c_i^0\right)$$

$$F^1 = \{e_j^i : j \in J, 1 \leq i \leq \alpha_j\} \cup \left(E^1 \setminus \cup_{i \in I} c_i^1\right)$$

where the v_j's, e_j^i's, and elements of E^0, E^1 are all distinct. We define

$$s_F, r_F : F^1 \to F^0$$

by, for all $j \in J, 1 \leq i \leq \alpha_j$,

$$s_F(e_j^i) = v_j = r_F(e_j^i)$$

and for all $e \in E^1 \setminus \cup_{i \in I} c_i^1$,

$$s_F(e) = \begin{cases} s(e) & \text{if } s(e) \notin c_i^0 \text{ for all } i \\ v_j & \text{if } s(e) \in c_i^0 \text{ and } [c_i] = [c_j] \text{ for some } j \in J \end{cases}$$

$$r_F(e) = \begin{cases} r(e) & \text{if } r(e) \notin c_i^0 \text{ for all } i \\ v_j & \text{if } r(e) \in c_i^0 \text{ and } [c_i] = [c_j] \text{ for some } j \in J. \end{cases}$$

Then

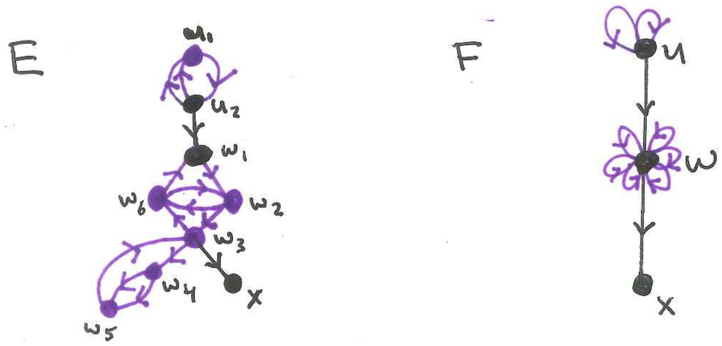
$$(M(E), \subseteq) \cong (M(F), \subseteq).$$

The folowing graph F is obtained from E by the method described above. In this case we have

$$M(E) = \{\{u_1, u_2\}, \{u_1, u_2, w_1, w_2, w_3, w_4, w_5, w_6\}, E^0\}$$

and

$$M(F) = \{\{u\}, \{u, w\}, \{u, w, x\}\}.$$



Proof. Define

$$\phi : E^0 \to F^0$$

by

$$\phi(v) = \begin{cases} v & \text{if} \quad v \notin c_i^0 \text{ for all } i \in I \\ v_j & \text{if} \quad v \in c_i^0 \text{ and } [c_i] = [c_j] \text{ for some } j \in J \end{cases}$$

Note that ϕ is well-defined since if $v \in c_i^0, c_k^0$ then c_i, c_k share a vertex, and thus $[c_i] = [c_k]$. Also note that ϕ is surjective, and that $\phi(v) = \phi(v')$ if and only if $v \geq v' \geq v$.

Now define

$$\Phi : 2^{E^0} \to 2^{F^0} \qquad \Phi(S) = \phi(S).$$

We claim that Φ restricts to give us the desired isomorphism.

First notice that, for any vertices $v, w \in E^0$, $v \geq w$ if and only if $\phi(v) \geq \phi(w)$. For the forward direction, observe that a path from v to w in E can be made into a path from $\phi(v)$ to $\phi(w)$ in F by simply removing any parts of the path which use part of a cycle. For the reverse direction, one can simply take the path from $\phi(v)$ to $\phi(w)$ and at any place the path passes through a v_j, just glue in edges from the appropriate cycle(s) in E to obtain a path from v to w.

Using this, for any $M \in M(E)$ it is straight-forward to check that $\Phi(M)$ satisfies (MT1), (MT2), and (MT3). Hence $\Phi(M)$ is a maximal tail in F and we can define

$$\hat{\Phi} : M(E) \to M(F)$$

$$M \mapsto \Phi(M).$$

$\hat{\Phi}$ is injective: Pick $M, N \in M(E)$ distinct. So, without loss of generality, there exists $v \in E^0$ such that $v \in M$, $v \notin N$. First suppose $v \notin c_i^0$ for all $i \in I$. Then v is the only pre-image of $\phi(v) = v$ under ϕ, and thus v lies in $\hat{\Phi}(M)$ but not in $\hat{\Phi}(N)$. Now assume $v \in c_i^0$ for some i. Suppose $\phi(v) \in \hat{\Phi}(N)$. Then we must have $w \in N$ where w lies on some c_j with $[c_j] = [c_i]$. But then necessarily $v \geq w$, and hence $v \in N$ by (MT1). This is a contradiction, and hence $\phi(v) \notin \hat{\Phi}(N)$. So in any case $\hat{\Phi}(N) \neq \hat{\Phi}(M)$.

$\hat{\Phi}$ is surjective: Fix $M \in M(F)$. Checking (MT1), (MT2), and (MT3) shows that the preimage $\phi^{-1}(M)$ is a maximal tail, and thus we have $\hat{\Phi}(\phi^{-1}(M)) = M$.

Therefore $\hat{\Phi}$ is a bijection, and also $N \subseteq M$ implies $\hat{\Phi}(N) \subseteq \hat{\Phi}(M)$. Conversely, suppose $\hat{\Phi}(N) \subseteq \hat{\Phi}(M)$. Taking preimages we have $\phi^{-1}(\hat{\Phi}(N)) \subseteq \phi^{-1}(\hat{\Phi}(M))$ and as in the previous paragraph each of $\phi^{-1}(\hat{\Phi}(N)), \phi^{-1}(\hat{\Phi}(M))$ is a maximal tail, so by injectivity of $\hat{\Phi}$ we have $\phi^{-1}(\hat{\Phi}(N)) = N$, $\phi^{-1}(\hat{\Phi}(M)) = M$. Hence $N \subseteq M$.

So $\hat{\Phi}$ is an isomorphism and we are done. $\square$

Finally, for any posets that do show up, their disjoint union will show up as well:

Proposition 3.5. *Let $\{E_i = (E_i^0, E_i^1, r_i, s_i) : i \in I\}$ be a nonempty collection of row-finite graphs with $E_i^0 \cap E_j^0 = \emptyset$ and $E_i^1 \cap E_j^1 = \emptyset$ for $i \neq j$. Let K be a field. Define*

$\bigsqcup_{i\in I} E_i$ *to be the graph* $\left(\bigsqcup_{i\in I} E_i^0, \bigsqcup_{i\in I} E_i^1, r, s\right)$*, where* $r(e) = r_i(e)$ *and* $s(e) = s_i(e)$ *for each* $e \in E_i^1$*. Then, as posets*

$$Spec_\gamma\left(L_K\left(\sqcup_{i\in I} E_i\right)\right) \cong \bigsqcup_{i\in I} Spec_\gamma(L_K(E_i))$$

and

$$Spec\left(L_K\left(\sqcup_{i\in I} E_i\right)\right) \cong \bigsqcup_{i\in I} Spec(L_K(E_i)).$$

Proof. Set $E = \bigsqcup_{i\in I} E_i$. For each $i \in I$, consider the algebra homomorphism

$$\pi_i : L_K(E) \to L_K(E_i)$$

which, for each $v \in E^0$, $e \in E^1$, satisfies

$$\pi_i(v) = \begin{cases} v & \text{if } v \in E_i^0 \\ 0 & \text{otherwise} \end{cases} \qquad \pi_i(e) = \begin{cases} e & \text{if } e \in E_i^1 \\ 0 & \text{otherwise} \end{cases} \qquad \pi_i(e^*) = \begin{cases} e^* & \text{if } e \in E_i^1 \\ 0 & \text{otherwise} \end{cases}$$

Note that π_i exists because $L_K(E), L_K(E_i)$ are generated by $E^0 \cup E^1 \cup (E^1)^*$, $E_i^0 \cup E_i^1 \cup (E_i^1)^*$, respectively, and the relations imposed on the generating set for $L_K(E)$ are all sent to zero by the map described above. Further, each π_i is surjective, and by Proposition 1.2, $ker(\pi_i) = \langle \sqcup_{j\neq i} E_j^0 \rangle$. We define maps

$$f : \bigsqcup_{i\in I} Spec(L_K(E_i)) \to Spec\left(L_K\left(E\right)\right)$$

$$f' : \bigsqcup_{i\in I} Spec_\gamma(L_K(E_i)) \to Spec_\gamma\left(L_K\left(E\right)\right)$$

by sending each prime ideal in $L_K(E_i)$ to its preimage under π_i. Then f, f' are injective, and for surjectivity pick J in either (1) $Spec(L_K(E))$ or (2) $Spec_\gamma(L_K(E))$. By Lemma 2.4 $E^0\backslash J$ is a maximal tail, so by (MT3) $E^0\backslash J \subseteq E_i^0$ for some $i \in I$. Hence $E_j^0 \subseteq J$ for all $j \neq i$, and hence $\langle \sqcup_{j\neq i} E_j^0 \rangle = ker(\pi_i) \subseteq J$. Therefore $J = \pi_i^{-1}(L)$ for some $L \in Spec(L_K(E_i))$. If J is graded so is L, and we obtain either (1) $f(L) = J$ or (2) $f'(L) = J$, accordingly.

So f, f' are bijections. From here it is clear that they are poset isomorphisms, so we are done. □

4. Special Sets of Vertices: $M(E)$ and $\mathcal{H}_E$

Recall from Chapter 2 that $(M(E), \supseteq) \cong (Spec_\gamma(L_K(E)), \subseteq)$, and that $M_\tau(E)$ together with K determine the structure of $Spec_\tau(L_K(E))$. Thus we take a moment to pause and delve into the structure of the maximal tails. We also make some notes on the structure of $\mathcal{H}_E$, due to its close ties with $M(E)$, and its importance in the theory of general two-sided ideals (e.g., $I \cap E^0 \in \mathcal{H}_E$ for *every* ideal I of $L_K(E)$).

Lemma 4.1. *Let E be countable. For each $v \in E^0$ define $N_v = \{w \in E^0 : w \geq v\}$. Then*

$$M(E) = \{N_v : v \in E^0 \text{ is a sink or there is a cycle } c \text{ in } E \text{ with } v \in c^0\}$$

$$\bigcup \{\cup_{i\in\mathbb{N}} N_{v_i} : v_1, v_2, ... \in E^0 \text{ with } v_i \gneqq v_{i+1} \text{ for all } i\}.$$

Proof. First observe that N_v, where $v \in E^0$ is either a sink or lies on a cycle, is a maximal tail. Also observe that $\bigcup_{i\in\mathbb{N}} N_{v_i}$ is a maximal tail whenever $v_1, v_2, ... \in E^0$ and $v_i \gneqq v_{i+1}$ for each i.

We now show every maximal tail is of one of these two forms. Pick $M \in M(E)$. Notice M is nonempty and countable, and label its elements v_i for $i \in \mathbb{N}$. Let j_1 be the smallest index used. Then $N_{v_{j_1}} \subseteq M$ by (MT1). If we have equality stop here. If not, let j be the smallest index for which $v_j \in M \backslash N_{v_{j_1}}$. Then, by (MT3) there exists $v_i \in M$ such that $v_j \geq v_i$, $v_{j_1} \geq v_i$. Set $j_2 = i$. We then have $N_{v_{j_1}} \subsetneqq N_{v_{j_2}}$ and $N_{v_{j_2}} \subseteq M$. If this second relation is an equality stop here. If not, let j be the smallest index for which $v_j \in M \backslash N_{v_{j_2}}$, and find $v_i \in M$ such that $v_j \geq v_i$, $v_{j_2} \geq v_i$. Set $j_3 = i$. Continue choosing j_i in this fashion. Either (1) eventually we find some n for which

$$N_{v_{jn}} = M$$

or (2) no such n exists, and we obtain

$$\bigcup_{i=1}^{\infty} N_{v_{j_i}} = M.$$

If we are in case (1), by (MT2) v_{j_n} is either a sink or lies on a cycle. If we are in case (2), then for the collection $\{v_{j_i} : i \in \mathbb{N}\}$ we have $v_{j_i} \gneq v_{j_{i+1}}$ for each i. In either case we are done. ☐

Lemma 4.2. *Suppose $M \in M_\tau(E)$. Then there is a cycle c in M with no exit in M, and*

$$M = \{v \in E^0 : v \geq w \text{ for some } w \in c^0\}.$$

Proof. First notice that since $M \in M_\tau(E)$ we are only guaranteed the existence of a closed simple path in M without any exits in M. But checking definitions shows that such a path is necessarily a cycle. Call this cycle c.

Set $N = \{v \in E^0 : v \geq w$ for some $w \in c^0\}$. By (MT1), we have $M \supseteq N$. Pick any $v \in M$ and let $y \in c^0 \subseteq M$. By (MT3), there exists $w \in M$ so that $v \geq w$, $y \geq w$. Let ρ be a path from y to w. Since $w \in M$, by (MT1) we know $\rho^0 \subseteq M$. But c has no exits in M, so ρ must only use edges from c^1. Hence $w \in c^0$, hence $v \in N$. So $M = N$ and we are done. ☐

Corollary 4.3. *Each $M \in M_\tau(E)$ contains a unique cycle with no exits in M, where we regard cycles having different basepoints but the same collection of edges as identical.*

Equivalently, whenever $M \in M_\tau(E)$, we have $C_{E^0 \setminus M} = \{c\}$ for a unique cycle c. From this point on we call this unique cycle $c_{E^0 \setminus M}$.

Although for a general $H \in \mathcal{H}_E$ we do not have that C_H contains only a single element, we do have that it only contains elements c_{H_i} where $E^0 \setminus H_i \in M_\tau(E)$. We take a slight detour from answering our main question to prove this, along with some results regarding the structure of general (not necessarily prime) ideals. We also use this chapter to emphasize the fundamental connection between $M(E)$ and $\mathcal{H}_E$.

Lemma 4.4. *If we define $m(E) := \{M \subseteq E^0 : M$ satisfies (MT1) and (MT2)$\}$ then*

$$m(E) = \left\{ \bigcup_{M \in N} M : N \subseteq M(E) \right\}.$$

Proof. Observe that for any $N \subseteq M(E)$, $\bigcup_{M \in N} M$ satisfies both (MT1) and (MT2), so we need only argue the other inclusion.

Pick $S \in m(E)$. If $S = \emptyset$ then $S = \bigcup_{M \in \emptyset} M$. Suppose $S \neq \emptyset$. Let $v \in S$ and set D to be the set of downward directed subsets of S containing v. Consider the poset $(D, \subseteq)$. We know D is nonempty as it contains $\{v\}$, and given a chain $\{X_i : i \in I\}$ in D we can see that it has an upper bound $\bigcup_{i \in I} X_i$ in D. Hence, by Zorn's Lemma D has a maximal element, say D_v. Notice

$$S = \bigcup_{v \in S} D_v$$

and further, each D_v is a maximal tail: since $D_v \in D$ we have $D_v \neq \emptyset$ and that (MT3) holds, and conditions (MT1) and (MT2) follow immediately from maximality. ☐

Corollary 4.5. *For any graph E,*

$$\mathcal{H}_E = \left\{ \bigcap_{M \in N} (E^0 \backslash M) : N \subseteq M(E) \right\}$$

where we set $\bigcap_{M \in \emptyset} (E^0 \backslash M) = E^0$.

Proof. By Lemma 1.3 we have

$$\mathcal{H}_E = \{H \subseteq E^0 : E^0 \backslash H \text{ satisfies (MT1) and (MT2)}\}.$$

The result then directly follows from Lemma 4.4. ☐

Corollary 4.6. *Let $H \in \mathcal{H}_E$, say $H = \cap_{i \in I} H_i$ with $E^0 \backslash H_i \in M(E)$ for each i. Then $C_H \subseteq \cup_{i \in I} C_{H_i}$.*

Proof. Note that if $H = E^0$, then $C_H = \emptyset$ so the result holds. So suppose $H \neq E^0$ and hence $I \neq \emptyset$. Suppose $c \in C_H$ so $c^0 \cap H = \emptyset$. Pick $v \in c^0$. Then $v \notin H$, so $v \notin H_i$ for some $i \in I$. So $v \in E^0 \backslash H_i$. Since $E^0 \backslash H_i$ is a maximal tail we in fact have $c^0 \subseteq E^0 \backslash H_i$, and hence $c^0 \cap H_i = \emptyset$. Let e be an exit of c. Then $r(e) \in H \subseteq H_i$, and therefore $c \in C_{H_i}$. ☐

Notice that combining this with Corollary 4.5 and reviewing the statement of Theorem 2.6 then gives a more detailed picture of the ideals of an arbitrary row-finite Leavitt Path Algebra.

Another consequence of Corollary 4.5 is as follows:

Corollary 4.7. *If E is row-finite, the set of graded ideals of $L_K(E)$ can be written*

$$\left\{ \bigcap_{I \in P} I : P \subseteq Spec_\gamma(L_K(E)) \right\}$$

where $\bigcap_{I \in \emptyset} I = L_K(E)$.

Proof. The intersection of graded ideals is a graded ideal so one inclusion is clear. For the other inclusion, pick a graded ideal J of $L_K(E)$. By Proposition 1.2 we know $J = \langle H_J \rangle$ for some $H_J \in \mathcal{H}_E$, and by Corollary 4.5 we know $H_J = \bigcap_{M \in N} (E^0 \backslash M)$ for some $N \subseteq M(E)$. Set $N' = \{E^0 \backslash M : M \in N\}$. Again by Proposition 1.2, since $\bigcap_{H \in N'} \langle H \rangle$ is graded we have

$$\bigcap_{H \in N'} \langle H \rangle = \langle K \rangle$$

for $K = \left(\bigcap_{H \in N'} \langle H \rangle\right) \cap E^0$. But each $H \in N'$ is saturated and hereditary so $\langle H \rangle \cap E^0 = H$ so in fact $K = \bigcap_{H \in N'} H$. Hence

$$\bigcap_{H \in N'} \langle H \rangle = \langle \bigcap_{H \in N'} H \rangle = \langle H_J \rangle = J.$$

Moreover, for each $H \in N'$, $E^0 \backslash H \in M(E)$, so by Lemma 2.1 $\langle H \rangle \in Spec_\gamma(L_K(E))$. □

5. The Non-Graded, Prime Ideals

First we prove a lemma showing that, for a polynomial $p(x)$ and a cycle c with $v \in c^0$, an ideal I of $L_K(E)$ contains $p(c_v)$ if and only if $I \supseteq p(c)$.

Lemma 5.1. *Let c be a cycle in E, and pick any vertices $v, w \in c^0$. Then, for any polynomial $p(x) \in K[x]$ and any ideal I of $L_K(E)$,*

$$p(c_v) \in I \Leftrightarrow p(c_w) \in I.$$

Proof. Without loss of generality $v \neq w$. First suppose $p(c_v) \in I$. Let γ be the portion of c_v which is a path from v to w, and let λ be the portion of c_w which is a path from w to v. Then $c_v = \gamma\lambda$ and $c_w = \lambda\gamma$ and hence, for any $n \in \mathbb{Z}_{\geq 0}$,

$$\gamma^* c_v^n \gamma = \gamma^* \underbrace{(\gamma\lambda)\cdots(\gamma\lambda)}_{n} \gamma = (\gamma^*\gamma) \underbrace{(\lambda\gamma)\cdots(\lambda\gamma)}_{n} = c_w^n.$$

Therefore

$$p(c_w) = \gamma^* p(c_v) \gamma \in I.$$

The other implication holds by a symmetric argument. □

Corollary 5.2. *Let E be row-finite. Suppose $I \in Spec_\tau(L_K(E))$ and $H = I \cap E^0$, $M = E^0 \backslash H$. Then*

$$I = \langle H \cup p(c_H) \rangle$$

for a polynomial $p(x) \in \mathcal{P}$ which is uniquely determined by I.

Proof. By Theorem 2.6 we know that $I = \langle H \cup P_C \rangle$ for a unique choice of $C \subseteq C_H, P = \{p_c(x) \in \mathcal{P} : c \in C\}$. Since $I \in Spec_\tau(L_K(E))$, Proposition 1.5 gives us $M \in M_\tau(E)$, and so applying Corollary 4.3 gives us $C_H = \{c_H\}$. Now $C \subseteq C_H$ and if $C = \emptyset$ then $P_C = \emptyset$. But I is non-graded, so we must have $C = \{c_H\}$. So $P_C = p(c_H)$ (and hence $P = \{p(x)\}$), where $p(x) \in \mathcal{P}$. □

Proposition 5.3. *Let E be row-finite. Let $M \in M_\tau(E)$, $H = E^0 \backslash M$, $p(x) \in \mathcal{P}$. Then $\langle H \cup p(c_H) \rangle \in Spec_\tau(L_K(E))$ if and only if $p(x)$ is irreducible.*

Proof. Set $I = \langle H \cup p(c_H) \rangle$ and $c = c_H$. Note that I is a non-graded, prime ideal if and only if $I/\langle H \rangle$ is a non-graded, prime ideal. Since $H \in \mathcal{H}_E$, by Lemma 1.8 we know $I/\langle H \rangle$ maps isomorphically to an ideal in $L_K(E/H)$, namely $\langle p(c) \rangle$. So without loss of generality, $I = \langle p(c) \rangle$ and $H = \emptyset$.

Fix any $v \in c^0$.

First we assume $I \in Spec_\tau(L_K(E))$.

Suppose $h \mid p$, so $p = gh$ for some $g, h \in K[x]$. Applying the isomorphism from Lemma 1.7 gives $p(c_v) = g(c_v)h(c_v)$ in $L_K(E)$. Consider any $\gamma \in L_K(E)$. Then

$$g(c_v)\gamma h(c_v) = g(c_v)v\gamma v h(c_v) = g(c_v)\gamma' h(c_v)$$

where $\gamma', g(c_v), h(c_v) \in vL_K(E)v$. By Lemma 1.7 we know that $vL_K(E)v$ is commutative, hence

$$g(c_v)\gamma h(c_v) = p(c_v)\gamma' \in I.$$

So $g(c_v)L_K(E)h(c_v) \subseteq I$. By primeness of I, this implies that either $g(c_v) \in I$ or $h(c_v) \in I$. Without loss of generality, suppose $g(c_v) \in I$, so

$$g(c_v) = \sum_{i=1}^{n} \alpha_i p(c_v) \beta_i$$

for some $\alpha_i, \beta_i \in L_K(E)$. Notice

$$g(c_v) = vg(c_v)v = \sum_{i=1}^{n} v\alpha_i v p(c_v) v \beta_i v$$

and hence, by Lemma 1.7, we have

$$g(x) = \sum_{i=1}^{n} a_i(x)p(x)b_i(x) = p(x)\sum_{i=1}^{n} a_i(x)b_i(x)$$

for appropriate $a_i(x), b_i(x) \in K[x, x^{-1}]$. If $a_i(x)b_i(x) \in K[x]$ for all i set $j = 0$. Otherwise set $-j$ to be the minimal degree of a term appearing in $\sum_{i=1}^{n} a_i(x)b_i(x)$. Then

$$g(x)x^j = g(x)h(x)d(x)$$

where

$$\sum_{i=1}^{n} a_i(x)b_i(x)x^j = d(x) \in K[x].$$

Since p is nonzero we know g is nonzero, and hence

$$x^j = h(x)d(x).$$

So $h(x) = \delta x^i$ for some $\delta \in K$ and $0 \leq i \leq j$. But $h(x)$ divides $p(x)$ and x^i does not divide $p(x)$ for any $i > 0$, so necessarily $h(x) = \delta$. Thus $h(x)$ is a unit, and therefore $p(x)$ is irreducible in $K[x]$.

Now assume $p(x)$ is irreducible.

Suppose A, B are ideals in $L_K(E)$ with $AB \subseteq I$. By Theorem 2.6 we can write

$$A = \langle H_A \cup P^A_{C_A} \rangle \qquad B = \langle H_B \cup P^B_{C_B} \rangle$$

for appropriate $H_A, H_B \in \mathcal{H}_E$, $C_A \subseteq C_{H_A}$, $C_B \subseteq C_{H_B}$, and $P^A, P^B \subseteq \mathcal{P}$. Now

$$H_A \cap H_B \subseteq AB \cap E^0 \subseteq I \cap E^0 = \emptyset,$$

where the last equality follows from Theorem 2.6 (notice $I = \varphi(\emptyset, \{c\}, \{p\})$). Suppose there exist $a \in H_A$, $b \in H_B$. Then $a, b \in E^0 = M$ and M is a maximal tail, so there exists some $w \in M$ such that $a \geq w$, $b \geq w$. Since H_A, H_B are hereditary this implies $w \in H_A \cap H_B$, which is a contradiction. So one of H_A, H_B is empty. By Lemma 1.9 we know $AB = BA$ so without loss of generality suppose $H_A = \emptyset$.

We claim that either $A \subseteq I$ or $B \subseteq I$. By Corollary 4.3 we have $C_A \subseteq C_{H_A} = \{c\}$. If $C_A = \emptyset$ then $A = 0$ and the claim holds. So suppose $C_A = \{c\}$ and hence $A = \langle p_A(c) \rangle$ for some $p_A(x) \in \mathcal{P}$.

Consider the case where $H_B = \emptyset$. So, as above, either $B = 0$ or $B = \langle p_B(c) \rangle$ for some $p_B(x) \in \mathcal{P}$. If $B = 0$ the claim holds, so suppose $B = \langle p_B(c) \rangle$. Then

$p_A(c_v)p_B(c_v) \in AB \subseteq \langle p(c) \rangle$, hence

$$p_A(c_v)p_B(c_v) = \sum_{i=1}^{n} \delta_i p(c_v)\gamma_i$$

for some $\delta_i, \gamma_i \in L_K(E)$. So we have

$$p_A(c_v)p_B(c_v) = \sum_{i=1}^{n} v\delta_i v p(c_v) v\gamma_i v = \sum_{i=1}^{n} \delta_i' p(c_v)\gamma_i'$$

for some $\delta_i', \gamma_i' \in vL_K(E)v$. Now applying Lemma 1.7 gives

$$p_A(x)p_B(x) = \sum_{i=1}^{n} f_i(x)p(x)g_i(x) = p(x)\sum_{i=1}^{n} f_i(x)g_i(x)$$

for some $f_i(x), g_i(x) \in K[x, x^{-1}]$. Since $p(x)$ is irreducible in $K[x]$ and not divisible by x, we know it is also irreducible in $K[x, x^{-1}]$. So either $p(x) \mid p_A(x)$ or $p(x) \mid p_B(x)$ in $K[x, x^{-1}]$. Then Lemma 1.7 gives either $p(c_v) \mid p_A(c_v)$ or $p(c_v) \mid p_B(c_v)$ in $L_K(E)$, and in either case the claim holds.

Now consider the case where $H_B \neq \emptyset$, so there exists some $w \in H_B$. Since $w \in E^0 = M$, by Lemma 4.2 and Corollary 4.3 we have $w \geq u$ for some $u \in c^0$. Hence $w \geq v$. Since H_B is hereditary we obtain $v \in H_B$, hence

$$p_A(c_v) = p_A(c_v)v \in AB \subseteq I,$$

hence $A \subseteq I$.

Therefore the claim holds and I is prime. Furthermore, by Proposition 1.2 and injectivity of φ in Theorem 2.6, I is non-graded and we are done. $\square$

We are now ready to prove the following theorem, as advertised in the beginning of the paper:

Theorem 5.4. *Let E be a row-finite graph and let K be a field. The map*

$$\Lambda : M_\tau(E) \times \dot{\mathcal{P}}(K) \to Spec_\tau(L_K(E))$$

$$(M, p) \mapsto \langle (E^0 \backslash M) \cup p(c_{E^0\backslash M})\rangle$$

is a bijection, where $c_{E^0\backslash M}$ *is the unique cycle in* $C_{E^0\backslash M}$*.*

Proof. By Proposition 5.3 we have that Λ is well-defined, and by Theorem 2.6 we have that Λ is injective. For surjectivity, pick $I \in Spec_\tau(L_K(E))$. By Corollary 5.2 we can write $I = \langle (E^0\backslash M) \cup p(c_{E^0\backslash M})\rangle$ for some $p \in \mathcal{P}(K)$, where $M = E^0\backslash I$. Since I is non-graded and prime, by Proposition 1.5 we know $M \in M_\tau(E)$ and by Proposition 5.3 we know p is irreducible. So $I = \Lambda(M, p)$. $\square$

Corollary 5.5. *Let E be a row-finite graph and let K be a field. Then*

$$Spec_\tau(L_K(E)) = \{\langle H \cup p(c_H)\rangle : H \subseteq E^0,\ E^0\backslash H \in M_\tau(E),\ p \in \dot{\mathcal{P}}(K)\}.$$

Moreover, the $\langle H \cup p(c_H)\rangle$ are distinct for distinct pairs (H, p), and $\langle H \cup p(c_H)\rangle \cap E^0 = H$.

We now record some results concerning the inclusion relations between the non-graded, prime ideals and the other prime ideals.

Lemma 5.6. *Let E be row-finite. If $I, J \in Spec_\tau(L_K(E))$ and $I \cap E^0 = J \cap E^0$, then $I \subseteq J$ if and only if $I = J$.*

Proof. Let $I, J \in Spec_\tau(L_K(E))$ with $H = I \cap E^0 = J \cap E^0$. By Corollary 5.2 and Proposition 5.3 we can write

$$I = \langle H \cup p(c_H)\rangle \qquad J = \langle H \cup p'(c_H)\rangle$$

for some $p, p' \in \dot{\mathcal{P}}$, and where $I = J$ if and only if $p = p'$. Since p, p' both have constant term 1, $p = p'$ if and only if p, p' are associates in $K[x]$, and since p, p' are irreducible, they are associates if and only if p' divides p. Now by [2, Proposition 2.8.8] we have that p' divides p if and only if $I \subseteq J$, which completes the proof. $\square$

Lemma 5.7. *Let E be row-finite and let $I \in Spec_\tau(L_K(E))$. Then the prime ideals of $L_K(E)$ properly containing I are precisely the prime ideals J such that $J \cap E^0$ properly contains $I \cap E^0$.*

Proof. Let J be a prime ideal of $L_K(E)$ with $H' = J \cap E^0$ properly containing $H = I \cap E^0$. From Corollary 5.5 we can write $I = \langle H \cup p(c_H) \rangle$ for some $p(x) \in \dot{\mathcal{P}}$, and $M = E^0 \backslash H \in M_\tau(E)$. Pick any $v \in H' \backslash H$. Then $v \in M$, so by Lemma 4.2 and Corollary 4.3 we know $v \geq w$ for one of the vertices $w \in c_H^0$. Since H' is hereditary we thus have $c_H^0 \subseteq H'$. So $c_H \subseteq \langle H' \rangle$, which means that $p(c_H) \subseteq \langle H' \rangle$. Hence $I \subsetneq \langle H' \rangle$. Since $\langle H' \rangle \subseteq J$, we therefore have $I \subsetneq J$.

Now suppose $J \in Spec(L_K(E))$ properly contains I. Then of course

$$J \cap E^0 \supseteq I \cap E^0.$$

Suppose $J \cap E^0 = I \cap E^0$. If J is non-graded, Lemma 5.6 immediately yields a contradiction. So we can assume J is graded. Then by Proposition 1.2

$$J = \langle J \cap E^0 \rangle = \langle I \cap E^0 \rangle$$

and hence $\langle I \cap E^0 \rangle \supsetneq I$. But $\langle I \cap E^0 \rangle \subseteq I$, so this is impossible.
Thus $J \cap E^0 \supsetneq I \cap E^0$. ☐

It can now be seen that our main question reduces to (1) What posets are of the form $(M(E), \supseteq)$? and (2) For each poset structure of $M(E)$ that is possible, what are the possible divisions of $M(E)$ into $M_\gamma(E)$ and $M_\tau(E)$?

6. The Partially Ordered Set of Prime Ideals

Now that we have a firm grasp on the structure of the ideals of $L_K(E)$ we use the remainder of the thesis to determine which posets we can make arise as $Spec(L_K(E))$, $Spec_\gamma(L_K(E))$, and $Spec_\tau(L_K(E))$.

First we recall a few relevant definitions. A partially ordered set $(A, \leq)$ has the descending chain condition (DCC) if it contains no infinite strictly decreasing chains. It has the ascending chain condition (ACC) if it contains no infinite strictly ascending chains. It *locally* has either property if all of its closed intervals have that property. We say an element $\alpha \in A$ covers an element $\beta \in A$ if $\alpha \geq \beta$ and whenever $x \in A$ with $\alpha \geq x \geq \beta$, we have either $x = \alpha$ or $x = \beta$.

We take the convention $0 \notin \mathbb{N}$.

Theorem 6.1. *Let $(A \preceq)$ be a countable poset that has the DCC and locally has the ACC, and let K be a field. There exists a countable, row-finite graph E such that*

$$(Spec(L_K(E)), \subseteq) = (Spec_\gamma(L_K(E)), \subseteq) \cong (A, \preceq).$$

For an explicit example of the construction we give in the following proof, see the image which appears immediately after the proof.

Proof. First set A^* to be the set of elements of A which cover infinitely many elements of A. For each $\alpha \in A^*$, fix a bijection

$$\varphi_\alpha : \{\beta \in A : \alpha \text{ covers } \beta\} \to \mathbb{N}.$$

Then we define

$$E^0 = \{v_\alpha^0 : \alpha \in A\} \cup \{v_\alpha^i : \alpha \in A^*,\ i \in \mathbb{N}\}$$

$$E^1 = \{e_\beta^\alpha : \alpha, \beta \in A,\ \alpha \text{ covers } \beta\} \cup \{f_\alpha, g_\alpha : \alpha \in A \backslash A^*\} \cup \{h_\alpha^i, k_\alpha^i : \alpha \in A^*,\ i \in \mathbb{N}\}$$

where all $v_\alpha^i, e_\beta^\alpha, f_\alpha, g_\alpha, h_\alpha^i$, and k_α^i are distinct. We also define maps $s, r : E^1 \to E^0$ by

$$s(e_\beta^\alpha) = \begin{cases} v_\alpha^0 & \text{if } \alpha \notin A^* \\ v_\alpha^{\varphi_\alpha(\beta)} & \text{if } \alpha \in A^* \end{cases} \qquad r(e_\beta^\alpha) = v_\beta^0$$

$$s(f_\alpha) = v_\alpha^0 = r(f_\alpha) \qquad s(g_\alpha) = v_\alpha^0 = r(g_\alpha)$$

$$s(h_\alpha^i) = v_\alpha^{i-1} = r(k_\alpha^i) \qquad r(h_\alpha^i) = v_\alpha^i = s(k_\alpha^i).$$

Put $E = (E^0, E^1, r, s)$ and note that E is countable and row-finite.

Claim 1: For all $\alpha, \beta \in A$, $\alpha \succeq \beta$ if and only if $v_\alpha^0 \geq v_\beta^0$.

First assume $\alpha \succeq \beta$. Since the interval $[\beta, \alpha]$ has the DCC and ACC, we know there exists a set $\{\alpha_1, ..., \alpha_n\} \subseteq A$ such that α covers α_1, α_i covers α_{i+1} for each i, and α_n covers β. Thus we have edges $e^{\alpha}_{\alpha_1}, e^{\alpha_1}_{\alpha_2}, ..., e^{\alpha_{n-1}}_{\alpha_n}, e^{\alpha_n}_{\beta}$. Hence

$$s(e^{\alpha}_{\alpha_1}) = v^{j_0}_{\alpha} \geq v^0_{\alpha_1} = r(e^{\alpha}_{\alpha_1}),\ v^{j_1}_{\alpha_1} \geq v^0_{\alpha_2},\ ...,\ v^{j_{n-1}}_{\alpha_{n-1}} \geq v^0_{\alpha_n}, v^{j_n}_{\alpha_n} \geq v^0_{\beta}$$

for some $j_i \in \mathbb{N} \cup \{0\}$. Now notice that for $v \in E^0$, $\alpha \in A^*$, $i \in \mathbb{N}$,

$$(*) \qquad v^i_{\alpha} \geq v \quad \Leftrightarrow \quad v^0_{\alpha} \geq v \quad \text{and} \quad v^i_{\alpha} \leq v \quad \Leftrightarrow \quad v^0_{\alpha} \leq v,$$

by concatenating paths with either $h^1_{\alpha} h^2_{\alpha} \cdots h^i_{\alpha}$ or $k^i_{\alpha} \cdots k^1_{\alpha}$. We thus have $v^0_{\alpha} \geq v^0_{\alpha_1}$, $v^0_{\alpha_1} \geq v^0_{\alpha_2}$, ..., $v^0_{\alpha_n} \geq v^0_{\beta}$, and hence $v^0_{\alpha} \geq v^0_{\beta}$.

Now assume $v^0_{\alpha} \geq v^0_{\beta}$, via some path ρ. Omit from ρ all edges f_{δ}, g_{δ}, as well as any edges at the beginning of ρ which precede the first e^{γ}_{δ}, and any at the end of ρ which succeed the last e^{γ}_{δ} to obtain a path $\hat{\rho}$. We can write

$$\hat{\rho} = E_1 H_1 E_2 H_2 \cdots H_{n-1} E_n$$

where each E_i is a path made of edges of the form e^{γ}_{δ}, and each H_i is a path of edges of the form $h^i_{\delta}, k^i_{\delta}$. For each i put

$$s(E_i) = v^{j_i}_{\alpha_i} \qquad r(E_i) = v^0_{\beta_i}$$

for some $\alpha_i, \beta_i \in A$ and $j_i \in \mathbb{N} \cup \{0\}$, so necessarily

$$s(H_i) = v^0_{\beta_i} \qquad r(H_i) = v^{j_{i+1}}_{\alpha_{i+1}}.$$

By the definition of s, r on the $h_{\delta_i}, k_{\delta_i}$ we in fact have $\beta_i = \alpha_{i+1}$. Also, by the construction of the e^{γ}_{δ}, from each E_i we get $\alpha_i \succeq \beta_i$. Hence

$$\alpha_i \succeq \alpha_{i+1}.$$

Hence $\alpha_1 \succeq \alpha_n \succeq \beta_n$. Now either $v^0_{\alpha} = s(\rho) = s(\hat{\rho}) = v^{j_1}_{\alpha_1}$, or there is a path between v^0_{α} and $v^{j_1}_{\alpha_1}$ made of $h^i_{\delta}, k^i_{\delta}$'s. In either case $\alpha = \alpha_1$. Similarly, either $v^0_{\beta} = r(\rho) =$

$r(\hat{\rho}) = v^0_{\beta_n}$, or there is a path between v^0_β and $v^0_{\beta_n}$ made of h^i_δ, k^i_δ's. In either case $\beta = \beta_n$. Therefore $\alpha \succeq \beta$ and the claim holds.

By Lemma 4.1 we can write

$$M(E) = \{N_v : v \in E^0 \text{ is a sink or there is a cycle } c \text{ in E with } v \in c^0\}$$

$$\bigcup \{\cup_{i\in\mathbb{N}} N_{v_i} : v_1, v_2, ... \in E^0 \text{ and } v_i \gneqq v_{i+1} \text{ for all } i\},$$

where $N_v = \{w \in E^0 : w \geq v\}$.

Claim 2: Any infinite sequence $v_1 \gneqq v_2 \gneqq \cdots$ in E^0 must eventually consist only of v^i_α's for some fixed $\alpha \in A^*$.

Suppose this is not the case for some sequence $\{v_i : i \in \mathbb{N}\}$. Then we have some

$$v^{i_1}_{\beta_1} \geq v^{i_2}_{\beta_2} \geq v^{i_3}_{\beta_3} \geq \cdots$$

with $\beta_i \neq \beta_{i+1}$ for all i, and hence, by $(*)$,

$$v^0_{\beta_1} \geq v^0_{\beta_2} \geq v^0_{\beta_3} \geq \cdots$$

Therefore

$$\beta_1 \succneqq \beta_2 \succneqq \beta_3 \succneqq \cdots$$

But this is impossible since A has the DCC. So the claim holds.

Thus for such a set $\{v_i : i \in \mathbb{N}\}$ we have

$$\cup_{i\in\mathbb{N}} N_{v_i} = \cup_{i\in\mathbb{N}} N_{v^i_\alpha} = N_{v^0_\alpha}$$

for some $\alpha \in A^*$ (note that $N_{v^i_\alpha} = N_{v^0_\alpha}$ for all $i \in \mathbb{N}$ by $(*)$). Also notice that every $v^0_\alpha \in E^0$ lies on a cycle (namely either f_α or $h^1_\alpha k^1_\alpha$). We thus have

$$M(E) = \{N_{v^0_\alpha} : \alpha \in A\}.$$

Note $N_{v_\alpha^0} \neq N_{v_\beta^0}$ for $\alpha \neq \beta$ by Claim 1.

Claim 3: $M(E) = M_\gamma(E)$.

Pick $M \in M(E)$ and consider any simple closed path c in M. Suppose c uses an edge of the form e_β^α. Then $\alpha \neq \beta$ and we obtain both $v_\alpha^i = s(e_\beta^\alpha) \geq r(e_\beta^\alpha) = v_\beta^0$ and $v_\alpha^i \leq v_\beta^0$ for some $i \in \mathbb{N} \cup \{0\}$. Hence by $(*)$ $v_\alpha^0 \geq v_\beta^0$ and $v_\alpha^0 \leq v_\beta^0$, and so by Claim 1 $\alpha \succeq \beta$ and $\alpha \preceq \beta$. This contradicts $\alpha \neq \beta$, so c must only use edges of the form $f_\alpha, g_\alpha, h_\alpha^i, k_\alpha^i$. If c uses an f_α then g_α is an exit for c in M, and if c uses a g_α then f_α is an exit for c in M. So suppose c only uses edges of the form h_α^i, k_α^i. Say $h_\alpha^i \in c^1$. Then either h_α^{i+1} or k_α^i is an exit for c, and by (MT1) we know all of the h_α^j, k_α^j's start and end in M, so again c has an exit in M. If $k_\alpha^i \in c^1$ then there is also some $h_\alpha^j \in c^1$, for one cannot make a cycle out of only k_α^i's. So by the previous argument c has an exit in M. This proves the claim.

Thus $M_\tau(E) = \emptyset$ so by Theorem 5.4, we have $Spec_\tau(L_K(E)) = \emptyset$. Hence

$$Spec(L_K(E)) = Spec_\gamma(L_K(E)).$$

Define the map

$$\Phi : A \to Spec_\gamma(L_K(E))$$
$$\alpha \mapsto \langle E^0 \backslash N_{v_\alpha^0} \rangle$$

which is well-defined and bijective by Lemma 2.1. Additionally Φ is an isomorphism: Fix $\alpha, \beta \in A$. Then we have the following equivalences:

$$\alpha \preceq \beta \quad \Leftrightarrow \quad v_\alpha^0 \leq v_\beta^0 \quad \Leftrightarrow \quad N_{v_\alpha^0} \supseteq N_{v_\beta^0} \quad \Leftrightarrow \quad E^0 \backslash N_{v_\alpha^0} \subseteq E^0 \backslash N_{v_\beta^0},$$

and further,

$$E^0 \backslash N_{v_\alpha^0} \subseteq E^0 \backslash N_{v_\beta^0} \quad \Leftrightarrow \quad \langle E^0 \backslash N_{v_\alpha^0} \rangle \subseteq \langle E^0 \backslash N_{v_\beta^0} \rangle$$

since, using the maps in Lemma 2.1:

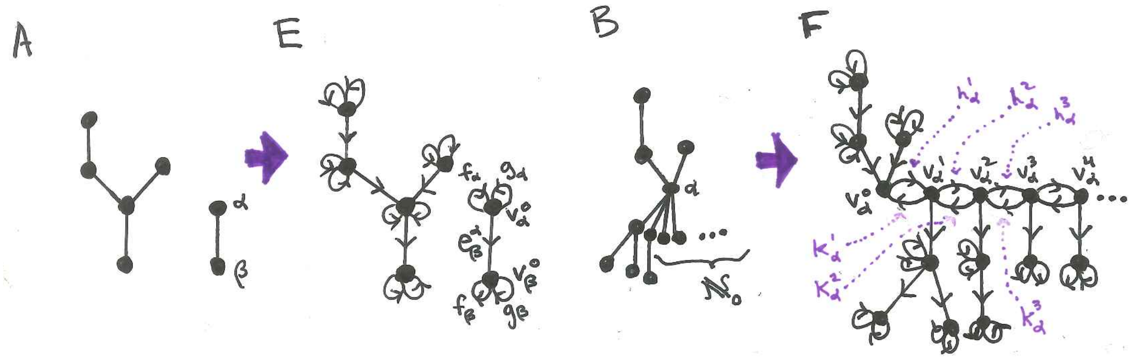
$$\langle E^0 \backslash N_{v^0_\alpha} \rangle \cap E^0 = E^0 \backslash \Psi(\langle E^0 \backslash N_{v^0_\alpha} \rangle) = E^0 \backslash \Psi(\Psi'(N_{v^0_\alpha})) = E^0 \backslash N_{v^0_\alpha}$$

and similarly

$$\langle E^0 \backslash N_{v^0_\beta} \rangle \cap E^0 = E^0 \backslash N_{v^0_\beta}.$$

This completes the proof. $\square$

For the posets A, B below, the construction in the above proof gives rise to the following graphs E, F:



Notice that, by the previous theorem, we find that any countable poset that locally has the ACC and DCC, and so that every chain has a lower bound, appears as both $Spec(L_K(E))$ and $Spec_\gamma(L_K(E))$ for some countable, row-finite graph E (as these conditions are stronger than the hypotheses of the theorem). We further note that the requirement for the chains to have lower bounds is also a necessary condition, by Lemmas 3.1 and 3.2.

Corollary 6.2. *Let $(A, \preceq)$ be a finite partially ordered set and let K be a field. There exists a finite graph E for which*

$$(Spec(L_K(E)), \subseteq) = (Spec_\gamma(L_K(E)), \subseteq) \cong (A, \preceq).$$

Proof. Using the proof of the previous theorem, since A is finite we have $A^* = \emptyset$, and hence the graph constructed in the proof has $|E^0| = |A|$ and $|E^1| \leq |A|^2 + 2|A|$. $\square$

7. On Finite Graphs

We now consider the posets of prime ideals which appear when we restrict our attention to finite graphs. We begin with the poset of prime graded ideals, where our answer falls out of Corollary 6.2.

Corollary 7.1. *The partially ordered sets which arise as $(Spec_\gamma(L_K(E)), \subseteq)$ for finite graphs E and arbitrary fields K are precisely the finite partially ordered sets.*

Proof. Let $(A, \preceq)$ be a finite poset. Pick a field K. By Corollary 6.2, there exists a finite graph E such that $(Spec_\gamma(L_K(E)), \subseteq) \cong (A, \preceq)$.

Now let E be some finite graph and K be some field. Since E^0 is finite, so is $M(E)$, and hence, by Lemma 2.1 so is $Spec_\gamma(L_K(E))$. So $(Spec_\gamma(L_K(E)), \subseteq)$ is a finite poset. □

Next we look at the poset of prime ideals of $L_K(E)$ for finite graphs E, after proving a lemma.

Lemma 7.2. *For any field K,*

$$|\dot{\mathcal{P}}(K)| = max\{\aleph_0, |K|\}.$$

Proof. First suppose K is finite. Notice that there are infinitely many monic, irreducible polynomials in $K[x]$, for otherwise we can list them as $f_1, ..., f_n$, say, and then see that $f_1 \cdots f_n + 1$ is also monic and irreducible, but is not on our list. The collection of monic irreducible polynomials in $K[x]$ is in bijection with $\dot{\mathcal{P}}$, so we obtain $|\dot{\mathcal{P}}| \geq \aleph_0$. Further note $|\dot{\mathcal{P}}| \leq |K[x]| = \aleph_0$.

Now suppose K is infinite. Notice $\{1 + \alpha x : \alpha \in K^*\} \subseteq \dot{\mathcal{P}} \subseteq K[x]$. Then, since $|\{1 + \alpha x : \alpha \in K^*\}| = |K|$ and $|\{f \in K[x] : deg(f) = n\}| = |K|$ for $n \geq 0$, and hence $|K[x]| = \aleph_0 \cdot |K| = |K|$, we obtain $\dot{\mathcal{P}} = |K|$. □

To assist in answering our question, we define a special type of partially ordered set. Take any poset $(B, \preceq)$ and any subset $\overline{B} \subseteq B$. Let U be any set. We define

$$A(B, \overline{B}, U) := (B \times \{0\}) \cup (\overline{B} \times U)$$

and give it a partial ordering $\leq$ defined by

$$(b, u) \leq (c, v) \Leftrightarrow (b, u) = (c, v), \text{ or } b \preceq c \text{ and } u = 0, \text{ or } b \precneqq c.$$

We note that it does not matter if U has an element labeled 0, since $A(B, \overline{B}, U \cup \{0\}) = A(B, \overline{B}, U \backslash \{0\})$.

Theorem 7.3. *The partially ordered sets which appear as $(Spec(L_K(E)), \subseteq)$ for finite graphs E and arbitrary fields K are precisely the posets $A(B, \overline{B}, U)$ where B is finite and U is infinite.*

Proof. Let B be some finite poset and U be some infinite set, and consider the poset $(A, \preceq)$ where $A = A(B, \overline{B}, U)$. As noted above, may assume U has no elements labeled 0.

STEP 1: We construct a finite graph E.

(For a visual of this construction, please see the image at the end of the proof.)

First note that there exists a field K with cardinality $|U|$. Fix some such K and notice that by Lemma 7.2 $|\dot{\mathcal{P}}(K)| = |U|$. Pick any bijection $\nu : U \to \dot{\mathcal{P}}(K)$. Now since B is finite, by Theorem 6.1 there exists a finite graph F such that

$$Spec(L_K(F)) = Spec_\gamma(L_K(F)) \cong B.$$

Let F be as in the proof of the theorem, but relabel each vertex v_b^0 as v_b.

From the proof of Theorem 6.1 the relation $\leq$ on F^0 is a partial order and we have a poset isomorphism

$$\varphi : F^0 \to B \qquad \varphi(v_b) = b.$$

Also

$$M(F) = M_\gamma(F) = \{N_{v_b} : b \in B\}$$

where

$$N_{v_b} = \{v \in F^0 : v \geq v_b\}$$

and $b \neq c$ implies $N_{v_b} \neq N_{v_c}$.

Now we define a new graph E by removing the edges g_b for $b \in \overline{B}$ from F. For each $b \in \overline{B}$ notice that N_{v_b} is again a maximal tail in E, and it contains the simple closed path f_b. Since $(f_b)^0 = \{v_b\}$ and $s^{-1}(v_b) \cap r^{-1}(N_{v_b}) = \{f_b\}$, the path f_b has no exit in N_{v_b}, and hence $N_{v_b} \in M_\tau(E)$.

We then have

$$M(E) = \{N_{v_b} : b \in B\} \qquad M_\tau(E) = \{N_{v_b} : b \in \overline{B}\}.$$

STEP 2: We construct a bijection Θ between A and $Spec(L_K(E))$.

First define

$$\Theta_1 : B \times \{0\} \to Spec_\gamma(L_K(E))$$

by

$$(b, 0) \mapsto \langle E^0 \backslash N_{v_b} \rangle.$$

Well-definedness and bijectiviy of Θ_1 follow immediately from Lemma 2.1.
Define

$$\Theta_2 : \overline{B} \times U \to Spec_\tau(L_K(E))$$

by

$$(b, u) \mapsto \langle (E^0 \backslash N_{v_b}) \cup \nu(u)(c_{E^0 \backslash N_{v_b}}) \rangle.$$

Since $\Theta_2(b,u) = \Lambda(N_{v_b}, \nu(u))$, where Λ is the bijection from Theorem 5.4, we find that Θ_2 is a bijection as well.

Glue Θ_1, Θ_2 together to obtain the bijection

$$\Theta : A \to Spec(L_K(E)).$$

STEP 3: We show Θ is an isomorphism.

Pick $(b,u),(c,v) \in A$ and put

$$\Theta(b,u) = I \qquad \Theta(c,v) = J.$$

We need to show that $(b,u) \preceq (c,v)$ if and only if $I \subseteq J$. Note that

$$I \cap E^0 = E^0 \backslash N_{v_b} \qquad J \cap E^0 = E^0 \backslash N_{v_c}$$

by Theorem 2.6.

Now notice the following equivalences:

$$\varphi(v_b) = b \preceq c = \varphi(v_c) \quad \Leftrightarrow \quad v_b \le v_c \quad \Leftrightarrow \quad N_{v_b} \supseteq N_{v_c} \quad \Leftrightarrow \quad I \cap E^0 \subseteq J \cap E^0,$$

$$u = 0 \quad \Leftrightarrow \quad I \text{ is graded},$$

and

$$b \neq c \quad \Leftrightarrow \quad I \cap E^0 \neq J \cap E^0.$$

So the condition for $(b,u) \preceq (c,v)$ is equivalent to

$$(*) \quad I = J, \text{ or} \quad I \cap E^0 \subseteq J \cap E^0 \text{ and } I \text{ is graded, or } I \cap E^0 \subsetneq J \cap E^0.$$

But $(*)$ holds if and only if $I \subseteq J$: If $I \cap E^0 \subseteq J \cap E^0$ and I is graded, $I = \langle I \cap E^0 \rangle \subseteq \langle J \cap E^0 \rangle \subseteq J$, and if $I \cap E^0 \subsetneq J \cap E^0$ and I is non-graded, by Lemma 5.7 we have $I \subseteq J$. Conversely, if $I \subseteq J$ then $I \cap E^0 \subseteq J \cap E^0$ and if $I \neq J$ and I is non-graded, by Lemma 5.7 we have $I \cap E^0 \subsetneq J \cap E^0$.

We therefore have $(b, u) \preceq (c, v)$ if and only if $I \subseteq J$.

Hence Θ gives us

$$(A, \preceq) \cong (Spec(L_K(E)), \subseteq),$$

and we are done with the first half of the proof.

Now pick any finite graph E and field K and consider $(Spec(L_K(E)), \subseteq)$. By Corollary 7.1, $Spec_\gamma(L_K(E))$ is finite. Put

$$B = Spec_\gamma(L_K(E)).$$

By Corollary 2.3 we can write $Spec_\gamma(L_K(E))$ as $\{\langle E^0 \backslash M\rangle : M \in M(E)\}$, so label the elements of B by b_M where $b_M = \langle E^0 \backslash M\rangle$. Take $\overline{B} = \{b_M : M \in M_\tau(E)\}$ and $U = \dot{\mathcal{P}}(K)$. Note that $\dot{\mathcal{P}}$ is infinite by Lemma 7.2.

We show the poset $(A, \preceq)$ for $A = A(B, \overline{B}, U)$ is isomorphic to $(Spec(L_K(E)), \subseteq)$.

STEP 4: We construct a bijection Σ between A and $Spec(L_K(E))$.

Define

$$\Sigma_1 : B \times \{0\} \to Spec_\gamma(L_K(E))$$

by

$$(b_M, 0) \mapsto \langle E^0 \backslash M\rangle$$

and define

$$\Sigma_2 : \overline{B} \times U \to Spec_\tau(L_K(E))$$

by

$$(b_M, u) \mapsto \langle (E^0 \backslash M) \cup u(c_{E^0 \backslash M})\rangle.$$

Notice $\Sigma_2(b_M, u) = \Lambda(M, u)$, where Λ is the bijection from Theorem 5.4 and so Σ_2 is a bijection.

Glue Σ_1 and Σ_2 together to get the bijection

$$\Sigma : A \to Spec(L_K(E)).$$

STEP 5: We show Σ is an isomorphism.

The details for this step are nearly identical to those in step 3, but we fill them in nonetheless.

Pick some $(b_M, u), (b_N, v) \in A$ and put

$$\Sigma(b_M, u) = I \qquad \Sigma(b_N, v) = J.$$

We show $(b_M, u) \preceq (b_N, v)$ if and only if $I \subseteq J$. Note that

$$I \cap E^0 = E^0 \backslash M \qquad J \cap E^0 = E^0 \backslash N$$

by Theorem 2.6.

We have

$$b_M = \langle I \cap E^0 \rangle \subseteq \langle J \cap E^0 \rangle = b_N \quad \Leftrightarrow \quad I \cap E^0 = \langle I \cap E^0 \rangle \cap E^0 \subseteq \langle J \cap E^0 \rangle \cap E^0 = J \cap E^0,$$

$$u = 0 \quad \Leftrightarrow \quad I \text{ is graded},$$

and

$$b_M \neq b_N \quad \Leftrightarrow \quad I \cap E^0 \neq J \cap E^0.$$

So the condition for which we have $(b_M, u) \preceq (b_N, v)$ is equivalent to $(*)$, above. But $(*)$ holds if and only if $I \subseteq J$, and we thus have $(b_M, u) \preceq (b_N, v)$ if and only if $I \subseteq J$.

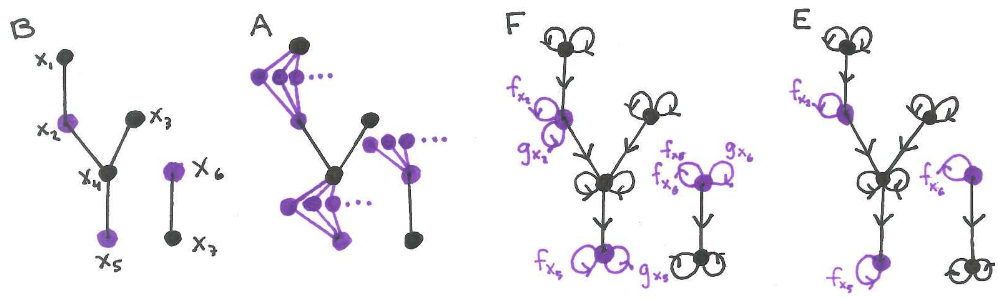
Hence

$$(A, \preceq) \cong (Spec(L_K(E)), \subseteq)$$

by Σ.

This completes the proof. $\square$

For an example of the construction given in Step 1, let B be as in the image below and take $\overline{B} = \{x_2, x_5, x_6\}$. The resulting poset $A = A(B, \overline{B}, U)$ and graphs F and E are then as follows, where each collection of dots which is followed by a "$\cdots$" is of cardinality $|U|$:



Corollary 7.4. *The partially ordered sets which arise as $(Spec_\tau(L_K(E)), \subseteq)$ for finite graphs E and arbitrary fields K are precisely the partially ordered sets $(V_n, \leq)$ where*

$$V_n = V \times \{1, ..., n\}$$

and $(p, i) \leq (q, j)$ if and only if $(p, i) = (q, j)$ or $i \precneqq j$, for some infinite set V, some $n \in \mathbb{Z}_{\geq 0}$, and some partial ordering $\preceq$ on $\{1, ..., n\}$.

Proof. Pick any such V, n, and $\preceq$ and consider $(V_n, \leq)$. Let $B = \{1, ..., n\}$, where $B = \emptyset$ if $n = 0$. Let $A = A(B, B, V)$ and notice

$$A \backslash (B \times \{0\}) \cong V_n.$$

From Theorem 7.3 there exists a finite graph E and some field K for which $Spec(L_K(E))$ $\cong A$ via a map Θ^{-1}. The restriction of Θ^{-1} to $Spec_\tau(L_K(E))$ gives an isomorphism of $Spec_\tau(L_K(E))$ onto $A \backslash (B \times \{0\})$. Hence

$$Spec_\tau(L_K(E)) \cong V_n.$$

Now pick any finite graph E and field K. By Theorem 7.3, there exists a partially ordered set $A = A(B, \overline{B}, V)$ with $|\overline{B}| = n$ finite and V infinite, such that $Spec(L_K(E)) \cong A$ via a map Σ. Notice Σ restricted to $A \backslash (B \times \{0\})$ gives

$$A \backslash (B \times \{0\}) \cong Spec_\tau(L_K(E)).$$

Further, if we label the elements of $\overline{B} = \{b_1, ..., b_n\}$ and denote the partial ordering on $\overline{B}$ by $\trianglelefteq$, and define $\preceq$ on $\{1, ..., n\}$ by $i \preceq j$ if and only if $b_i \trianglelefteq b_j$, we have $V_n \cong A \backslash (B \times \{0\})$. So $V_n \cong Spec_\tau(L_K(E))$.

$\square$

References

[1] G. Abrams, *Leavitt path algebras: the first decade,* Bull. Math. Sci. **5** (2015) 59-120.

[2] G. Abrams, P. Ara, M. Siles Molina, *Leavitt Path Algebras,* Springer (TBD).

[3] P. Ara, M.A. Moreno, E. Pardo, *Nonstable K-theory for graph algebras,* Algebras Represent. Theory **10** (2007) 157-178.

[4] G. Aranda Pino, E. Pardo, M. Siles Molina, *Exchange Leavitt path algebras and stable rank,* J. Algebra **305** (2006) 912-936.

[5] G. Aranda Pino, E. Pardo, M. Siles Molina, *Prime Spectrum and Primitive Leavitt Path Algebras,* Indiana Univ. Math. J. **58** (2009) 869-890.

[6] K. Goodearl, *Leavitt Path Algebras and Direct Limits,* Contemporary Mathematics **480** (2009) 165-187.

[7] I. Kaplansky, *Commutative Rings,* The University of Chicago Press, Chicago (1974).